JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **94**, 116–136 (1983)

The Existence of Local Semiflows for Nonlinear Differential Equations in Banach Spaces, and Applications to Partial Differential Equations

RICHARD A. GRAFF

The MITRE Corporation, Bedford, Massachusetts 01730

Submitted by G.-C. Rota

INTRODUCTION

Considerable success has been achieved in recent years in the investigation of abstract quasilinear differential equations in Banach spaces, and the results applied to the proof of existence and uniqueness of solutions to various systems of quasilinear parabolic and hyperbolic partial differential equations. However, an abstract theory applicable to truly nonlinear equations has until now eluded investigators.

In the present article we propose one way to approach nonliner existence theory by presenting an abstract theorem which can be applied to yield existence, uniqueness, and well-posedness results for many purely nonlinear partial differential equations and systems of equations. The basic idea behind the technique is to assume that additional variables may be introduced into the abstract equation, and an additional equation in the same variables constructed, in such a way that singular nonlinear terms in the original equation become bounded terms in the first equation of a two-equation system in which the singular terms of the second equation are quasilinear. The resulting system may be investigated by quasilinar techniques and the results applied to the original equation via our abstract theorem, which gives sufficient conditions for the existence of a one-to-one correspondence between solutions to the system and solutions to the original equation.

The author developed the idea for the abstract correspondence theorem after reading two remarkable papers [1, 2] of Dorroh in which the existence of solutions to certain systems of first- and second-order nonlinear partial differential equations of evolution type is established by directly manipulating the equations to derive quasilinear systems whose solutions are shown to correspond under certain conditions to solutions of the original systems. It seemed to this author that Dorroh's techniques hinted at the existence of a powerful and general abstract principle. This speculation

0022-247X/83 $3.00

provided the impetus for the investigation which has resulted in the present article.

The Correspondence Theorem is developed in Section 2 and is applied to four nonlinear partial differential equations and systems of equations in Section 3. The first application, to nonlinear first-order scalar-valued equations in R^n, is a special case of the first-order symmetric hyperbolic systems treated in [1], and is included here primarily to illustrate an uncomplicated application of the correspondence theorem. Two approaches are presented to treat this example. The first is basically that developed in [1], though refinements made possible by the correspondence theorem enable us to show existence of a local flow for the equation in $H^s(R^n, R)$ for $s > n/2 + 2$, an improvement over the $s > n/2 + 3$ required in [1]. The second application, to nonlinear parabolic equations of arbitrary order, uses a modification of the method developed by Dorroh in [2] to treat nonlinear second-order parabolic equations. The third application shows the existence of local flows for nonlinear second-order wave equations in R^n, and the fourth application establishes the existence of a global flow for a generalized Korteweg–de Vries equation to which the quasilinear existence theory of Kato [9, 10] is not applicable. Two methods are also presented to treat the fourth application.

For this degree of generality in the nonlinearities involved, the existence of solutions to the equations treated in Applications 3 and 4 is established for the first time in this paper. The existence of solutions in Application 2 is also a new result for nonlinear equations of order greater than two, and for second-order equations the results developed here improve and correct the results in [2]. In addition, our methods show that, for each solution $\alpha = \alpha(t, x)$, there exists $\delta > 0$ such that $\alpha(\cdot, \cdot)$ does not lose any derivatives with respect to the state space variables for $0 \leqslant t < \delta$, which is not shown in [1, 2].

Since the correspondence theorem applies to abstract differential equations in Banach spaces, solutions to partial differential equations obtained via this theorem are globally defined with respect to the space variables in the domain of the equation, but usually only locally defined with respect to the time variable. However, as in the case of the fourth application, it is possible under some circumstances to obtain solutions which are globally defined with respect to both space and time parameters.

Although the correspondence theorem can be generalized to time-dependent differential equations in Banach spaces, for most applications this generalization will not be necessary. If X is a Banach space, time-dependent differential equations in X may be regarded as time-independent differential equations in $X \times R$, and treated with the version of the correspondence theorem developed in this article.

1. Preliminaries

We first establish the form of the abstract differential equations we will deal with and the definitions of solutions and semiflows. If Y and Z are Banach spaces, Y a linear subspace of Z such that the inclusion of Y in Z is continuous and dense, then a differential equation in (Y, Z) is a continuous map from an open subset of Y to Z.

DEFINITION 1. If U is open in Y, $g\colon U \to Z$ a differential equation in (Y, Z), then a curve $\alpha\colon [0, T) \to U$ will be called a *solution* to the differential equation (integral curve for the vector field) if $\alpha(\cdot) \in C^0([0, T), Y) \cap C^1([0, T), Z)$ and $\dot{\alpha}(t) = g(\alpha(t))$ for $t \in [0, T)$.

DEFINITION 2. If $g\colon U \to Z$ is a differential equation in (Y, Z), V is open in $[0, \infty) \times U$, and $F(\cdot, \cdot)\colon V \to Y$ is continuous, define $F(t)$ for each $t \geqslant 0$ by $F(t)(y) = F(t, y)$. We will call F a (*local*) *semiflow* for $g(\cdot)$ if

(1) $F(0) = Id_U$.

(2) $F(t) \circ F(s)(y) = F(t+s)(y)$ whenever the right side of the equality is defined.

(3) Each $F(\cdot, y)$ is an integral curve for g.

In the application section, we will make use of the Sobolev spaces $L_s^q(R^n, R^{\tilde{n}})$ and the uniformly local Sobolev spaces $L_{s,\mathrm{ul}}^q(R^n, R^{\tilde{n}})$, $s \in [0, \infty]$. If $q = 2$, we will denote Sobolev spaces by $H^s(R^n, R^{\tilde{n}})$. We will assume that the reader is familiar with the properties of Sobolev spaces, and for uniformly local Sobolev spaces we refer the reader to the discussion in [8]. We will need the following three relations between Sobolev spaces, uniformly local Sobolev spaces, and the space of bounded uniformly continuous maps from R^n to $R^{\tilde{n}}$:

(1) For each $s \geqslant 0$, $L_s^q(R^n, R^{\tilde{n}}) \subset L_{s,\mathrm{ul}}^q(R^n, R^{\tilde{n}})$.

(2) For $s > n/q$, $l \in [0, s]$, $t \in [0, s-l]$, pointwise multiplication induces a continuous bilinear map

$$L_{s-l,\mathrm{ul}}^q(R^n, L(R^m, R^{\tilde{n}})) \times L_{t+l}^q(R^n, R^m) \to L_t^q(R^n, R^{\tilde{n}}).$$

(3) For $s > n/q$, $L_s^q(R^n, R^{\tilde{n}}) \subset C_{\mathrm{uc}}^0(R^n, R^{\tilde{n}})$, the space of bounded uniformly continuous maps from R^n to $R^{\tilde{n}}$.

Finally, we need to define the class of nonlinear maps which comprise allowable arguments in applications to partial differential equations. For applications to equations defined on compact manifolds we can dispense with any restrictions and consider all smooth nonlinear arguments of the state space, the solutions, and derivatives of the solutions. However, for applications to equations defined on R^n we must restrict our attention to nonlinear arguments which behave sufficiently well near infinity to induce continuous nonlinear maps between appropriately chosen Sobolev spaces, and which have the property that the associated nonlinear maps obtained in applying the correspondence theorem also induce continuous maps between appropriate Sobolev spaces.

DEFINITION 3. Let $q \in (1, \infty)$, $a(\cdot, \cdot)$: $R^n \times R^m \to^{C^\infty} R^k$. We will say that $a(\cdot, \cdot)$ is a type $\mathscr{S}^q$ map if $a(\cdot, 0) \in L^q_\infty(R^n, R^k)$ and if, for each $i > 0$,

(1) $D_1^i a(\cdot, 0)$: $R^n \to L^i(R^n, R^k)$ is an element of $L^q_\infty(R^n, L^i(R^n, R^k))$, where $D_1^i a(\cdot, \cdot)$ denotes the ith partial derivative of $a(\cdot, \cdot)$ with respect to the first variable.

(2) $D^i a(R^n \times B)$ is a bounded subset of $L^i(R^n \times R^m, R^k)$ for each bounded subset B of R^m.

If $q = 2$, we will omit the superscript and simply refer to $a(\cdot, \cdot)$ as a type $\mathscr{S}$ map.

2. THE CORRESPONDENCE THEOREM

Let X, Y, Z, Y_J, Z_J, V_1, and V_2 be Banach spaces with $Y \subset X \subset Z$, $Y_J \subset Z_J$, $V_1 \subset V_2$, such that each inclusion is linear, continuous, and dense. Assume that A: $Z \to Z_J$ is a continuous linear map such that $A(Y) \subset Y_J$, $Y = \{x \in X\colon Ax \in Y_J\}$, and the induced map $Id \times A$: $Y \to X \times Y_J$ is a topological embedding. Let $\tilde{Y} = (Id \times A)(Y)$.

Let W be open in $X \times Y_J$, and define U by $U = (Id \times A)^{-1}(W)$. Let f: $W \to Z$ be a continuous map, and define g: $U \to Z$ by $g = f \circ (Id \times A)$. Note that U is open in Y and that $g(\cdot)$ is a differential equation in (Y, Z). Let $\tilde{f}$: $W \to Z$ and h: $W \to Z_J$ be continuous maps, and let H: $W \to Z \times Z_J$ be the differential equation in $(X \times Y_J,\ Z \times Z_J)$ defined by $H(x, p) = (\tilde{f}(x, p), h(x, p))$. Let p_i defined on $Z \times Z_J$ be projection onto the ith factor.

THEOREM. *Assume that $\tilde{f}|_{\tilde{Y}} = f|_{\tilde{Y}}$. Assume in addition that there exists a continuous linear map $l\colon Z \times Z_J \to V_2$ and a continuous map $F\colon W \to L(V_1, V_2)$ such that*

(1) $l(X \times Y_J) \subset V_1$,

(2) $\text{Kernel}(l) = \{(z_1, z_2) \in Z \times Z_J\colon Az_1 = z_2\}$,

(3) *For each* $(x, p) \in W$, $l(H(x, p)) = F(x, p)\, l(x, p)$,

(4) *For each integral curve $\sigma(\cdot)$ in W for $H(\cdot, \cdot)$, the time-dependent linear Cauchy problem $(d\rho/dt)(\cdot) = F(\sigma(\cdot))\,\rho(\cdot)$, $\rho(0) = 0$, has only the trivial*

solution $\rho(\cdot) \equiv 0$. *Then, for each* $y_0 \in U$, *the map* $\alpha(\cdot) \mapsto (\alpha(\cdot), A\alpha(\cdot))$ *establishes a one-to-one correspondence between solutions to* $g(\cdot)$ *with initial value* y_0 *and solutions to* $H(\cdot, \cdot)$ *with initial value* (y_0, Ay_0).

Proof. For each $y \in U$, note that $l(f(y, Ay), h(y, Ay)) = l(\tilde{f}(y, Ay), h(y, Ay)) = F(y, Ay)\ l(y, Ay) = F(y, Ay)(0) = 0$. Let $\alpha(\cdot)$ be a solution to $g(\cdot)$ with $\alpha(0) = y_0$. Then $A\alpha(\cdot)$ is continuously differentiable in Z_J, and for each $t_0 \geqslant 0$,

$$\begin{aligned} \frac{d}{dt} A \circ \alpha(t_0) &= A \frac{d\alpha}{dt}(t_0) = Ag(\alpha(t_0)) \\ &= Af(\alpha(t_0), A\alpha(t_0)) = h(\alpha(t_0), A\alpha(t_0)), \end{aligned}$$

which implies that

$$\begin{aligned} \frac{d}{dt}(\alpha(\cdot), A\alpha(\cdot))(t_0) &= (\tilde{f}(\alpha(t_0), A\alpha(t_0)), h(\alpha(t_0), A\alpha(t_0))) \\ &= H(\alpha(t_0), A\alpha(t_0)). \end{aligned}$$

Conversely, assume that $\beta(\cdot)$ is a solution to $H(\cdot, \cdot)$ with $\beta(0) = (y_0, Ay_0)$, and define $\alpha(\cdot) = p_1 \circ \beta(\cdot)$. Clearly $\alpha(\cdot)$ is continuous in X. We must show that $\alpha(\cdot)$ is a continuous curve in U and that $(d\alpha/dt)(\cdot) = g(\alpha(\cdot))$.

Define $\rho(\cdot) = l \circ \beta(\cdot)$. Then $\rho(\cdot)$ is a continuous curve in V_1 and is continuously differentiable in V_2. Furthermore, $(d\rho/dt)(\cdot) = l(d\beta/dt)(\cdot) = l(H(\beta(\cdot))) = F(\beta(\cdot))\rho(\cdot)$, and $\rho(0) = l(\beta(0)) = l(y_0, Ay_0) = 0$. Since this linear Cauchy problem has only one solution, it follows that $\rho(\cdot) \equiv 0$.

Since Kernel$(l) = \{(z_1, z_2) \in Z \times Z_J : Az_1 = z_2\}$, it follows that $\beta = (\alpha, A\alpha)$. But $\alpha(\cdot)$ is a curve in X and $A\alpha(\cdot)$ is a curve in Y_J, which implies that α is a curve in Y. The continuity of $(\alpha(\cdot), A\alpha(\cdot))$ in $\tilde{Y}$ implies the continuity of $\alpha(\cdot)$ in Y. So it only remains to note that $(d\alpha/dt)(\cdot) = p_1 \circ (d\beta/dt)(\cdot) = p_1 \circ H(\beta(\cdot)) = \tilde{f}(\beta(\cdot)) = \tilde{f}(\alpha(\cdot), A\alpha(\cdot)) = f(\alpha(\cdot), A\alpha(\cdot)) = g(\alpha(\cdot))$. ∎

COROLLARY 1. *If integral curves for* $H(\cdot, \cdot)$ *are unique, then integral curves for* $g(\cdot)$ *are unique.*

COROLLARY 2. *If there exists a semiflow for* $H(\cdot, \cdot)$, *then there exists a semiflow for* $g(\cdot)$.

In applications of the correspondence theorem to partial differential equations, it will often be the case that $f(\cdot, \cdot)$ is a continuous map from $X \times Y_J$ to X, since the object of establishing the correspondence between $g(\cdot)$ and $H(\cdot, \cdot)$ is to reduce the nonlinear differential equation to a nonsingular term in a quasilinear system.

Note that the most important spaces in the statement of the correspondence theorem are X and Y_J. The spaces Z and Z_J can be chosen somewhat arbitrarily, subject to the requirement that each integral curve for $H(\cdot, \cdot)$ be continuously differentiable in $Z \times Z_J$. It is possible that intermediate spaces Z_1, Z_2 may exist, with $X \times Y_J \subset Z_1 \times Z_2 \subset Z \times Z_J$, such that integral curves for $H(\cdot, \cdot)$ are continuously differentiable in $Z_1 \times Z_2$. Similarly, an intermediate space $\tilde{Z}$ may exist, with $Y \subset \tilde{Z} \subset Z$, such that any semiflow for $g(\cdot)$ in (Y, Z) is automatically also a semiflow for $g(\cdot)$ in $(Y, \tilde{Z})$.

The obvious differential equation to associate with $g(\cdot)$ is the system $(f(\cdot, \cdot), Af(\cdot, \cdot))$. It is trivial (and is implied as a special case of the above theorem) that solutions to $g(\cdot)$ with initial value y_0 and solutions to $(f(\cdot, \cdot), Af(\cdot, \cdot))$ with initial value (y_0, Ay_0) are in one-to-one correspondence. Furthermore, for partial differential equations, $(f(\cdot, \cdot), Af(\cdot, \cdot))$ turns out to be a quasilinear system. Unfortunately, quasilinear existence and uniqueness theory is usually not applicable to this particular system. The significance of the correspondence theorem is that it suggests elementary modifications which may be made to the system $(f(\cdot, \cdot), Af(\cdot, \cdot))$ without changing those solutions whose initial values are in $\tilde{Y} = (Id \times A)(Y)$. To find a suitable system $H(\cdot, \cdot)$ to associate with $g(\cdot)$, first compute $Af(\cdot, \cdot)$, then make linear modifications to $(f(\cdot, \cdot), Af(\cdot, \cdot))$ until a system is obtained to which quasilinear existence theory can be applied, subject to the restriction that each modification must leave the value of the equation unchanged on $\tilde{Y} = (Id \times A)(Y)$.

Remark. Assume that $H(\cdot, \cdot)$, l, and $F(\cdot, \cdot)$ have been defined which satisfy all of the conditions of the theorem except for condition (4).

(i) If $V_1 = V_2$, so each operator $F(u, p)$ is a bounded linear operator, then condition (4) is automatically satisfied.

(ii) The uniqueness of solutions to the time-dependent linear Cauchy problem $(dp/dt)(\cdot) = F(\sigma(\cdot))\, p(\cdot)$, $p(0) \in V_1$, can be demonstrated under more general conditions than are necessary to prove existence of solutions. For example, it follows from [3, Proposition 3.5] that a sufficient condition to guarantee uniqueness of solutions is the existence of a locally bounded nonnegative function $\vartheta(\cdot)$ on W such that each $F(w) \in L(V_1, V_2)$ admits an extension to the generator of a strongly continuous linear semigroup of operators on V_2 of type $(1, \vartheta(w))$ (it is not necessary that the extensions be unique). A nonlinear analogue of this proposition also exists [4] which establishes the uniqueness of solutions to nonlinear differential equations in Banach spaces under more general conditions than those covered by current existence theory.

3. Applications

3.1. *The General First-Order Scalar-Valued Equation in R^n*

Let $a(\cdot,\cdot)$: $R^n \times R^{n+1} \to^{C^\infty} R$ be a function of type $\mathscr{S}$, and consider the equation

$$\frac{\partial u}{\partial t}(t,x) = a\left(x, u(t,x), \frac{\partial u}{\partial x_1}(t,x), \dots, \frac{\partial u}{\partial x_n}(t,x)\right) \quad \text{in} \quad R^{n+1},$$

where $x = (x_1, \dots, x_n) \in R^n$.

Let $s > n/2 + 1$, $Y_J = (H^s(R^n, R))^n$, $Y = H^{s+1}(R^n, R)$, $Z_J = (H^{s-1}(R^n, R))^n$, $Z = X = H^s(R^n, R)$, $A = (\partial/\partial x_1, \dots, \partial/\partial x_n)$. For $n > 1$, let $V_1 = (H^1(R^n, R))^{n+n^2}$, $V_2 = (H^0(R^n, R))^{n+n^2}$. For $n = 1$, let $V_1 = V_2 = H^0(R, R)$. Let $W = X \times Y_J$, and define f: $X \times Y_J \to X$ by $f(u, p)(x) = a(x, u(x), p(x))$, where $p = (p^1, \dots, p^n) \in R^n$. Let $\tilde{f} = f$. Then, for each $1 \leqslant i \leqslant n$,

$$\lambda_i \circ A \circ f(u,p)(x) = a_i(x, u(x), p(x)) + a_{n+1}(x, u(x), p(x))\, u_i(x) + \sum_{j=1}^{n} a_{j+n+1}(x, u(x), p(x))(p^j)_i\,(x),$$

where λ_i denotes projection onto the ith factor and the subscripts in the terms on the right side denote partial derivatives with respect to the indicated coordinates.

For $1 \leqslant i \leqslant n$, define $g_i(u, p)$ by $g_i(u,p)(x) = a_i(x, u(x), p(x))$, and for $0 \leqslant i \leqslant n$, define $f_i(u,p)$ by $f_i(u,p)(x) = a_{i+n+1}(x, u(x), p(x))$. Let $g = (g_1, \dots, g_n)$. Then the assumption that $a(\cdot,\cdot)$ is of type $\mathscr{S}$ implies that g_i: $X \times Y_J \to H^s(R^n, R)$ and f_i: $X \times Y_J \to H^s_{\mathrm{ul}}(R^n, R) \subset L(H^s(R^n, R), H^s(R^n, R))$.

If $(u(\cdot), Au(\cdot))$ is a solution to (f, Af) in Y, then along this integral curve, $u_i = p^i$ and $(p^i)_j = (p^j)_i$. So it is reasonable to replace u_i with p^i and $(p^j)_i$ with $(p^i)_j$. Define $h(u,p) \in Z_J$ by

$$\lambda_i \circ h(u,p) = g_i(u,p) + f_0(u,p)\, p^i + \sum_{j=1}^{n} f_j(u,p)(p^i)_j.$$

Then the system $H(\cdot,\cdot) = (f(\cdot,\cdot), h(\cdot,\cdot))$ is clearly a nonhomogeneous quasilinear symmetric first-order hyperbolic system in $n+1$ real variables, since we have

$$\frac{\partial u}{\partial t} = f(u,p),$$

$$\frac{\partial p}{\partial t} = \sum_{j=1}^{n} f_j(u,p)\frac{\partial p}{\partial x_j} + (g(u,p) + f_0(u,p)\,p).$$

Thus Kato's quasilinear existence theory [9] implies that integral curves for $H(\cdot, \cdot)$ are unique and that $H(\cdot, \cdot)$ generates a local semiflow (in fact, a local flow).

For each $1 \leqslant i \leqslant n$,

$$\lambda_i(Af(u, p) - h(u, p)) = f_0(u, p)(u_i - p^i) + \sum_{j=1}^{n} f_j(u, p)((p^j)_i - (p^i)_j).$$

Thus, for $n = 1$, we let $V_1 = V_2 = H^0(R, R)$, $l(u, p) = Au - p$. Define $F(u, p)(v) = f_0(u, p)\, v$. Since each $F(u, p)$ is a bounded linear operator, condition (4) of the theorem is trivially satisfied.

For $n > 1$, let $\{\lambda_i \colon 1 \leqslant i \leqslant n, \lambda_{ij} \colon 1 \leqslant i, j \leqslant n\}$ be the coordinate projections on $(H^r(R^n, R))^{n+n^2}$ and define $l(u, p)$ as follows:

$$\lambda_i \circ l(u, p) = u_i - p^i, \qquad \lambda_{i,j} \circ l(u, p) = (p^i)_j - (p^j)_i.$$

Let $(u(\cdot), p(\cdot))$ be a solution to $H(\cdot, \cdot)$. Differentiating $(p^i)_j - (p^j)_i$ along the solution, we get

$$\begin{aligned}
\frac{d}{dt}((p^i)_j - (p^j)_i) = {} & a_{m+n+1}(\partial/\partial x_m)((p^i)_j - (p^j)_i) + a_{n+1}((p^i)_j - (p^j)_i) \\
& + (a_{n+1,i} + a_{n+1,n+1} u_i + a_{n+m+1,n+1}(p^i)_m)(u_j - p^j) \\
& - (a_{n+1,j} + a_{n+1,n+1} u_j + a_{n+m+1,n+1}(p^j)_m)(u_i - p^i) \\
& + (a_{n+m+1,n+1} p^j + a_{n+m+1,j} \\
& \quad + a_{q+n+1,m+n+1}(p^q)_j)((p^i)_m - (p^m)_i) \\
& - (a_{n+m+1,n+1} p^i + a_{n+m+1,i} \\
& \quad + a_{q+n+1,m+n+1}(p^q)_i)((p^j)_m - (p^m)_j),
\end{aligned}$$

where the subscripts denote partial derivatives with respect to the indicated coordinates and the summation convention is employed that if a given index appears in both terms of a product, then that product is assumed to be summed over all values of that index between 1 and n (and $a_{i,j}(t)(x) = a_{i,j}(x, u(t)(x), p(t)(x))$).

So, for $v \in V_1 = (H^1(R^n, R))^{n+n^2}$, define $F(u, p)(v) \in V_2$ by

$$\begin{aligned}
\lambda_i \circ F(u, p)(v) &= a_{n+1} v_i + a_{n+j+1} v_{ji}, \\
\lambda_{i,j} \circ F(u, p)(v) &= a_{m+n+1}(\partial/\partial x_m)\, v_{ij} + a_{n+1} v_{ij} \\
&\quad + (a_{n+1,i} + a_{n+1,n+1} u_i + a_{n+m+1,n+1}(p^i)_m)\, v_j \\
&\quad - (a_{n+1,j} + a_{n+1,n+1} u_j + a_{n+m+1,n+1}(p^j)_m)\, v_i \\
&\quad + (a_{n+m+1,n+1} p^j + a_{n+m+1,j}
\end{aligned}$$

$$+ a_{q+n+1,m+n+1}(p^q)_j)\, v_{im}$$
$$- (a_{n+m+1,n+1}\, p^i + a_{n+m+1,i}$$
$$+ a_{q+n+1,m+n+1}(p^q)_i)\, v_{jm},$$

where $v_i = \lambda_i \circ v$ and $v_{i,j} = \lambda_{i,j} \circ v$.

The first term in the expression for $\lambda_{i,j} \circ F(u, p)$ is the only term which is unbounded. It follows from a well-known result in [12] that $F(\cdot, \cdot)$ satisfies condition (ii) of the remark of the previous section. The correspondence theorem thus implies that the nonlinear equation $\dot{u} = f(u, Au)$ generates a local flow in $H^{s+1}(R^n, R)$.

Remark. (i) Let $n > 1$ in this example and let s, Y, Y_J, X, and A be as defined above. A more sophisticated approach than the one taken above would be to select $\tilde{f}(\cdot, \cdot)$ so that the expression for $A\tilde{f} - h$ is less complicated than the expression for $Af - h$. For instance, we could let $Z = H^{s-1}(R^n, R)$, $Z_J = (H^{s-2}(R^n, R))^n$, $h(\cdot, \cdot)$ as defined above, and define $\tilde{f}(\cdot, \cdot)$ by $\tilde{f}(u, p) = f(u, p) + f_i(u, p)(u_i - p^i)$. In this case it suffices to let $V_1 = (H^1(R^n, R))^n$, $V_2 = (H^0(R^n, R))^n$, and $l(u, p) = Au - p$. Verification for this equation of the details of this method is left to the reader. However, detailed verification will be provided for higher-order analogues of this method in the next two applications.

(ii) The arguments given here carry over without modification to the case of nonlinear first-order symmetric hyperbolic systems (see [1]).

3.2. *Parabolic Systems in R^n*

Let $m \in 2N$ and let $a(\cdot, \cdot)\colon R^n \times R^{\tilde{n}+\tilde{m}} \to R^{\tilde{n}}$ be a map of type $\mathscr{S}^q$, where $\tilde{m} = \tilde{n}(\sum_{j=1}^{m} \binom{n+j-1}{j})$. Consider the equation

$$\frac{\partial u}{\partial t}(t, x) = a\left(x, u(t, x), \frac{\partial u}{\partial x_1}(t, x), \ldots, \frac{\partial u}{\partial x_n}(t, x), \ldots, \frac{\partial^m u}{\partial x_1 \cdots \partial x_1}(t, x), \ldots, \frac{\partial^m u}{\partial x_n \cdots \partial x_n}(t, x)\right)$$

in $R^{n+\tilde{n}}$, where $x = (x_1, \ldots, x_n) \in R^n$, $u(t, x) \in R^{\tilde{n}}$. Let $\{\lambda_{i_1, \ldots, i_k}\colon 1 \leqslant k \leqslant m,\ 1 \leqslant i_1 \leqslant \cdots \leqslant i_k \leqslant n\}$ be the $R^{\tilde{n}}$-valued projections on $R^{\tilde{m}}$. Let $q \in (1, \infty)$, $s \geqslant 2m$, and assume that $s > n/q$. Let

$$X = L_s^q(R^n, R^{\tilde{n}}),$$
$$Y_J = (L_s^q(R^n, R^{\tilde{n}}))^{\tilde{m}}, \qquad Y = L_{s+m}^q(R^n, R^{\tilde{n}}),$$
$$Z = L_{s-m}^q(R^n, R^{\tilde{n}}), \qquad Z_J = (L_{s-2m}^q(R^n, R^{\tilde{n}}))^{\tilde{m}},$$
$$V_1 = (L_m^q(R^n, R^{\tilde{n}}))^{\tilde{m}}, \qquad V_2 = (L^q(R^n, R^{\tilde{n}}))^{\tilde{m}},$$

$$A = \left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n},\frac{\partial^2}{\partial x_1 \partial x_1},\dots,\frac{\partial^m}{\partial x_1 \cdots \partial x_1},\dots,\frac{\partial^m}{\partial x_n \cdots \partial x_n}\right),$$

$$l(u, p) = Au - p.$$

If $p \in R^{\tilde{m}}$, let $p^{i_1,\dots,i_k} = \lambda_{i_1,\dots,i_k}(p)$.

We assume that the operator on the right side of the equation is locally uniformly strongly elliptic, by which we mean that, for each bounded set $B \subset R^{\tilde{n}+\tilde{m}}$, there exists a constant $c = c(B) > 0$ such that

$$\left(\sum_{1 \leqslant j_1 \leqslant \cdots \leqslant j_m \leqslant n} (-1)^{m/2+1} \frac{\partial a(x, w)}{\partial p^{j_1,\dots,j_m}} \xi_{j_1}\xi_{j_2} \cdots \xi_{j_m} v, v\right) > c\,|\xi|^m\,|v|^2$$

for each $(x, w) \in R^n \times B$, $\xi \in R^n$, $v \in R^{\tilde{n}}$ (note that each partial derivative in the above expression is an $\tilde{n} \times \tilde{n}$ matrix). This implies that, for each $y \in Y$, the derivative at y of the map induced by the right side of our nonlinear equation extends uniquely to the generator of an analytic linear semigroup on $L_r^q(R^n, R^{\tilde{n}})$ for $0 \leqslant r \leqslant s$.

For $(u, p) \in X \times Y_J$, define $f(u, p) \in X$ by $f(u, p)(x) = a(x, u(x), p(x))$, and let $f_{p^{j_1,\dots,j_k}}(u, p)$ denote the function defined by

$$f_{p^{j_1,\dots,j_k}}(u, p)(x) = \frac{\partial a(x, u(x), p(x))}{\partial p^{j_1,\dots,j_k}}.$$

We continue use of the standard summation convention employed in the previous example.

We first compute $A \circ f$. For each $1 \leqslant k < m$, $1 \leqslant i_1 \leqslant \cdots \leqslant i_k \leqslant n$, $\lambda_{i_1,\dots,i_k} \circ A \circ f$ is a polynomial differential operator of weight and order k (for an exposition of polynomial differential operators, see [10, Chap. 16]). It follows that

$$\lambda_{i_1,\dots,i_k} \circ A \circ f(u, p) = f_{p^{j_1,\dots,j_m}}(u, p)\frac{\partial^k p^{j_1,\dots,j_m}}{\partial x_{i_1} \cdots \partial x_{i_k}} + \tilde{h}_{i_1,\dots,i_k}(u, p),$$

where $\tilde{h}_{i_1,\dots,i_k}$ is of weight and order k, but of order $(k-1)$ in each $p^{j_1,\dots,j_m}$. Define $h_{i_1,\dots,i_k}(u, p)$ by replacing each $\partial^k p^{j_1,\dots,j_m}/\partial x_{i_1} \cdots \partial x_{i_k}$ with $\partial^m p^{i_1,\dots,i_k}/\partial x_{j_1} \cdots x_{j_m}$ in the above expression. For $k = m$, we have

$$\begin{aligned}\lambda_{i_1,\dots,i_m} \circ A \circ f(u, p) = {} & f_u(u, p)\frac{\partial^m u}{\partial x_{i_1} \cdots \partial x_{i_m}} \\ & + \sum_{1 \leqslant k \leqslant m} f_{p^{j_1,\dots,j_k}}(u, p)\frac{\partial^m p^{j_1,\dots,j_k}}{\partial x_{i_1} \cdots \partial x_{i_m}} \\ & + \tilde{\tilde{h}}_{i_1,\dots,i_m}(u, p),\end{aligned}$$

where $\tilde{\tilde{h}}_{i_1,\ldots,i_m}$ is of weight m and order $(m-1)$. Define $h_{i_1,\ldots,i_m}(u,p)$ by replacing each $\partial^m p^{j_1,\ldots,j_k}/\partial x_{i_1}\cdots\partial x_{i_m}$ with $\partial^k p^{i_1,\ldots,i_m}/\partial x_{j_1}\cdots\partial x_{j_k}$ and $\partial^m u/\partial x_{i_1}\cdots\partial x_{i_m}$ with $p^{i_1,\ldots,i_m}$ in the expression for $\lambda_{i_1,\ldots,i_m}\circ A\circ f(u,p)$; define $\tilde{h}_{i_1,\ldots,i_m}$ by

$$h_{i_1,\ldots,i_m}(u,p)=f_{p^{j_1,\ldots,j_m}}(u,p)\frac{\partial^m p^{i_1,\ldots,i_m}}{\partial x_{j_1}\cdots\partial x_{j_m}}+\tilde{h}_{i_1,\ldots,i_m}(u,p).$$

Define $h(\cdot,\cdot)$ and $\tilde{h}(\cdot,\cdot)$ by $\lambda_{i_1,\ldots,i_k}\circ h(u,p)=h_{i_1,\ldots,i_k}(u,p)$ and $\lambda_{i_1,\ldots,i_k}\circ\tilde{h}(u,p)=\tilde{h}_{i_1,\ldots,i_k}(u,p)$. Then $h(\cdot,\cdot)$ is of weight and order m, $\tilde{h}(\cdot,\cdot)$ is of weight m and order $(m-1)$, and

$$h(u,p)=f_{p^{j_1,\ldots,j_m}}(u,p)\frac{\partial^m p}{\partial x_{j_1}\cdots\partial x_{j_m}}+\tilde{h}(u,p).$$

Define $\tilde{f}\colon X\times Y_J\to Z$ and $\tilde{\tilde{f}}\colon X\times Y_J\to Z$ by

$$\tilde{f}(u,p)=f(u,p)+\sum_{k=1}^{m}f_{p^{j_1,\ldots,j_k}}(u,p)\left(\frac{\partial^k u}{\partial x_{j_1}\cdots\partial x_{j_k}}-p^{j_1,\ldots,j_k}\right),$$

$$\tilde{\tilde{f}}(u,p)=f(u,p)+\sum_{k=1}^{m-1}f_{p^{j_1,\ldots,j_k}}(u,p)\left(\frac{\partial^k u}{\partial x_{j_1}\cdots\partial x_{j_k}}-p^{j_1,\ldots,j_k}\right)$$
$$-f_{p^{j_1,\ldots,j_m}}(u,p)\,p^{j_1,\ldots,j_m}.$$

Define $\tilde{H}(u,p)=(\tilde{\tilde{f}}(u,p),\tilde{h}(u,p))$. Then

$$H(u,p)=f_{p^{j_1,\ldots,j_m}}(u,p)\frac{\partial^m(u,p)}{\partial x_{j_1}\cdots\partial x_{j_m}}+\tilde{H}(u,p).$$

Since $H(\cdot,\cdot)$ is of weight m and order $(m-1)$, it follows from [11, Theorem 16.10] that $\tilde{H}(\cdot,\cdot)$ extends to a map from $L^r_{s-1}(R^n,R^{\bar{n}})\times(L^r_{s-1}(R^n,R^{\bar{n}}))^{\bar{m}}$ to $Z\times Z_J$, where $r=qs/(s-1)$. It also follows (from the Sobolev embedding theorems) that $L^q_{s-\varepsilon}(R^n,R^{\bar{n}})\times(L^q_{s-\varepsilon}(R^n,R^{\bar{n}}))^{\bar{m}}\subset L^r_{s-1}(R^n,R^{\bar{n}})\times(L^r_{s-1}(R^n,R^{\bar{n}}))^{\bar{m}}$ for $\varepsilon=1-n/qs$. Since $s-\varepsilon>n/q$, each $f_{p^{j_1,\ldots,j_m}}(\cdot,\cdot)$ extends to a smooth map from $L^q_{s-\varepsilon}(R^n,R^{\bar{n}})\times(L^q_{s-\varepsilon}(R^n,R^{\bar{n}}))^{\bar{m}}$ to $L(Z\times Z_J, Z\times Z_J)$. Thus $H(\cdot,\cdot)$ satisfies the hypotheses of the parabolic quasilinear existence theory developed by Sobolevskii [13], which implies that there exists a unique integral curve for $H(\cdot,\cdot)$ beginning at each point of $X\times Y_J$. While Sobolevskii did not show the continuous dependence of these integral curves on the initial data, this author has developed a proof that the integral curves do depend continuously on the initial data. The assumptions needed to establish continuous dependence are more restrictive than those made by Sobolevskii, but they do apply to $H(\cdot, \cdot)$. The proof of continuous dependence will appear in [5].

To complete the proof that the equation $\dot{u} = f(u, Au)$ generates a local semigroup in Y, it suffices to show that $A\tilde{f} - h$ satisfies the condition of the correspondence theorem.

Let $t \in [m, s)$, $t > n/q$. Note that, for each $1 \leqslant j, k \leqslant m$,

$$\begin{aligned}
&\frac{\partial^j}{\partial x_{i_1} \cdots \partial x_{i_j}} (f_{p^{j_1,\ldots,j_k}}(u, p)(u_{j_1,\ldots,j_k} - p^{j_1,\ldots,j_k})) \\
&\quad = f_{p^{j_1,\ldots,j_k}}(u, p) \left(\frac{\partial^j u_{j_1,\ldots,j_k}}{\partial x_{i_1} \cdots \partial x_{i_j}} - \frac{\partial^j p^{j_1,\ldots,j_k}}{\partial x_{i_1} \cdots \partial x_{i_j}}\right) \\
&\quad\quad + g_{i_1,\ldots,i_j}^{j_1,\ldots,j_k}(u, p)(u_{j_1,\ldots,j_k} - p^{j_1,\ldots,j_k}),
\end{aligned}$$

where $g_{i_1,\ldots,i_j}^{j_1,\ldots,j_k}(u, p)$ is a (finite) sum of terms of the form

$$g^{k_1,\ldots,k_l}(u, p) \frac{\partial^{j-l}}{\partial x_{k_{l+1}} \cdots \partial x_{k_j}} \qquad \text{for} \quad 1 \leqslant l \leqslant j,$$

where $g^{k_1,\ldots,k_l}(\cdot, \cdot)$ in turn is a smooth map from $L_t^q(R^n, R^{\tilde{n}}) \times (L_t^q(R^n, R^{\tilde{n}}))^{\tilde{m}}$ to $L_{t-l}^q(R^n, L(R^{\tilde{n}}, R^{\tilde{n}}))$ induced by a monomial differential operator of weight l and order between 1 and l in the derivatives of u and p (and $(k_1,\ldots, k_j)$ is a permutation of $(i_1,\ldots, i_j)$).

Thus $g^{k_1,\ldots,k_l}(\cdot, \cdot)(\partial^{j-l}/\partial x_{k_{l+1}} \cdots \partial x_{k_j})$ is a smooth map from $\tilde{X} \times \tilde{Y}_J$ to $L(L_{j-\omega}^q(R^n, R^{\tilde{n}}),\ L^q(R^n, R^{\tilde{n}}))$, where $\tilde{X} = L_t^q(R^n, R^{\tilde{n}})$, $\tilde{Y}_J = (L_t^q(R^n, R^{\tilde{n}}))^{\tilde{m}}$, and $\omega = \min\{t - n/q, l\}$. It follows that there exists $\delta > 0$ such that $g_{i_1,\ldots,i_j}^{j_1,\ldots,j_k}(\cdot, \cdot)$ is a smooth map from $\tilde{X} \times \tilde{Y}_J$ to $L(L_{m-\delta}^q(R^n, R^{\tilde{n}}), L^q(R^n, R^{\tilde{n}}))$.

For each $1 \leqslant j < m$, $1 \leqslant i_1 \leqslant \cdots \leqslant i_j \leqslant n$,

$$\begin{aligned}
&\lambda_{i_1,\ldots,i_j}(A \circ \tilde{f}(u, p) - h(u, p)) \\
&\quad = \sum_{k=1}^{m} \frac{\partial^j}{\partial x_{i_1} \cdots \partial x_{i_j}} (f_{p^{j_1,\ldots,j_k}}(u, p)(u_{j_1,\ldots,j_k} - p^{j_1,\ldots,j_k})) \\
&\quad\quad + f_{p^{j_1,\ldots,j_m}}(u, p) \left(\frac{\partial^j p^{j_1,\ldots,j_m}}{\partial x_{i_1} \cdots \partial x_{i_j}} - \frac{\partial^m p^{i_1,\ldots,i_j}}{\partial x_{j_1} \cdots \partial x_{j_m}}\right) \\
&\quad = \sum_{k=1}^{m-1} \frac{\partial^j}{\partial x_{i_1} \cdots \partial x_{i_j}} (f_{p^{j_1,\ldots,j_k}}(u, p)(u_{j_1,\ldots,j_k} - p^{j_1,\ldots,j_k})) \\
&\quad\quad + f_{p^{j_1,\ldots,j_m}}(u, p) \frac{\partial^m}{\partial x_{j_1} \cdots \partial x_{j_m}} (u_{i_1,\ldots,i_j} - p^{i_1,\ldots,i_j})
\end{aligned}$$

$$+\sum_{l=1}^{j}\left(\frac{\partial^l}{\partial x_{i_1}\cdots\partial x_{i_l}}(f_{pj_1,\ldots,j_m}(u,p)\right)$$
$$\times\frac{\partial^{j-l}}{\partial x_{i_{l+1}}\cdots\partial x_{i_j}}(u_{j_1,\ldots,j_m}-p^{j_1,\ldots,j_m})$$
$$=f_{pj_1,\ldots,j_m}(u,p)\frac{\partial^m}{\partial x_{j_1}\cdots\partial x_{j_m}}(u_{i_1,\ldots,i_j}-p^{i_1,\ldots,i_j})$$
$$+g_{i_1,\ldots,i_j}(u,p)(Au-p),$$

where $g_{i_1,\ldots,i_j}\colon \tilde{X}\times\tilde{Y}_J\to^{C^\infty} L(L^q_{m-\delta}(R^n,R^{\tilde{n}})^{\tilde{m}}, L^q(R^n,R^{\tilde{n}}))$ for some $\delta>0$. A similar calculation shows that this decomposition also holds for $j=m$. Thus we conclude that

$$A\circ\tilde{f}(u,p)-h(u,p)$$
$$=f_{pj_1,\ldots,j_m}(u,p)\frac{\partial^m}{\partial x_{j_1}\cdots\partial x_{j_m}}(Au-p)+g(u,p)(Au-p),$$

where $g\colon\tilde{X}\times\tilde{Y}_J\to^{C^\infty} L((L^q_{m-\delta}(R^n,R^{\tilde{n}}))^{\tilde{m}}, V_2)$ for some $\delta>0$.

Note that each $f_{pj_1,\ldots,j_m}(\cdot,\cdot)$ extends to a smooth map from $\tilde{X}\times\tilde{Y}_J$ to $L(L^q(R^n,R^{\tilde{n}}), L^q(R^n,R^{\tilde{n}}))$. Furthermore, each integral curve for $H(\cdot,\cdot)$ is locally uniformly Holder continuous in $\tilde{X}\times\tilde{Y}_J$ with Holder exponent $\gamma=\min\{(s-t)/m, 1\}$. Thus, if $\sigma(\cdot)$ is an integral curve for $H(\cdot,\cdot)$ in $\tilde{X}\times\tilde{Y}_J$, then Sobolevskii's theory of time-dependent linear parabolic evolution equations [13] implies that the Cauchy problem

$$\dot{\rho}(t)=f_{pj_1,\ldots,j_m}(\sigma(t))\frac{\partial^m}{\partial x_{j_1}\cdots\partial x_{j_m}}\rho(t)+g(\sigma(t))(\rho(t)),\qquad \rho(0)=0,$$

has only the constant solution $\rho(\cdot)\equiv 0$ in V_2.

Remark. The above result is considerably weaker than the best possible. For instance, using an analogous technique to the one employed in the remark in Application 4, it is possible to extend the proof of local flow existence and integral curve uniqueness to $L^q_{s+m}(R^n,R^{\tilde{n}})$ for $q>n/m$ and $s>m-1+n/qm$ (verification depends upon technical results which will be developed in [5]), and it may be possible to extend this result to still lower values of s.

3.3. *Second-Order Wave Equations in R^n*

Let $a(\cdot,\cdot)\colon R^n\times R^{2\tilde{n}+m}\to R^{\tilde{n}}$ be a map of type $\mathscr{S}$, where $m=\tilde{n}(2n+n(n+1)/2)$. Consider the equation

$$\frac{\partial^2 u}{\partial t^2}(t,x) = a\left(x, u(t,x), \frac{\partial u}{\partial t}(t,x), \frac{\partial u}{\partial x_1}(t,x),\ldots, \frac{\partial u}{\partial x_n}(t,x),\right.$$
$$\left.\frac{\partial^2 u}{\partial t\,\partial x_1}(t,x),\ldots, \frac{\partial^2 u}{\partial t\,\partial x_n}(t,x), \frac{\partial^2 u}{\partial x_1\,\partial x_1}(t,x),\ldots, \frac{\partial^2 u}{\partial x_n\,\partial x_n}(t,x)\right)$$

in $R^{n+\tilde{n}}$, where $x = (x_1,\ldots, x_n) \in R^n$, $u(t,x) \in R^{\tilde{n}}$. We transform this equation into an equivalent first-order system in the usual way by writing it as follows:

$$\frac{\partial u}{\partial t}(t,x) = v(t,x),$$

$$\frac{\partial v}{\partial t}(t,x) = a\left(x, u(t,x), v(t,x), \frac{\partial u}{\partial x_1}(t,x),\ldots, \frac{\partial u}{\partial x_n}(t,x),\right.$$
$$\left.\frac{\partial v}{\partial x_1}(t,x),\ldots, \frac{\partial v}{\partial x_n}(t,x), \frac{\partial^2 u}{\partial x_1\,\partial x_1}(t,x),\ldots, \frac{\partial^2 u}{\partial x_n\,\partial x_n}(t,x)\right).$$

Let $\{\lambda_i, v_i, \lambda_{i,j}\colon 1 \leqslant i \leqslant j \leqslant n\}$ be the $R^{\tilde{n}}$-valued projections on R^m. Let $\tilde{m} = m + \tilde{n}n(n+1)/2$, so that $R^{\tilde{m}} = R^m \times R^{\tilde{n}n(n+1)/2}$, and let $\{v_{i,j}\colon 1 \leqslant i \leqslant j \leqslant n\}$ be the additional projections on $R^{\tilde{m}}$. If $v \in R^{\tilde{m}}$, let $p^i = \lambda_i(v)$, $q^i = v_i(v)$, $p^{i,j} = \lambda_{i,j}(v)$, $q^{i,j} = v_{i,j}(v)$. Assume that $s > n/2 + 4$, and let $Y = H^{s+2}(R^n, R^{\tilde{n}}) \times H^{s+1}(R^n, R^{\tilde{n}})$. Let

$$Y_J = (H^s(R^n, R^{\tilde{n}}) \times H^{s-1}(R^n, R^{\tilde{n}}))^{n+n(n+1)/2},$$
$$X = H^s(R^n, R^{\tilde{n}}) \times H^{s-1}(R^n, R^{\tilde{n}}),$$
$$Z = H^{s-1}(R^n, R^{\tilde{n}}) \times H^{s-2}(R^n, R^{\tilde{n}}),$$
$$Z_J = (H^{s-3}(R^n, R^{\tilde{n}}) \times H^{s-4}(R^n, R^{\tilde{n}}))^{n+n(n+1)/2},$$
$$V_1 = (H^2(R^n, R^{\tilde{n}}) \times H^1(R^n, R^{\tilde{n}}))^{n+n(n+1)/2},$$
$$V_2 = (H^1(R^n, R^{\tilde{n}}) \times H^0(R^n, R^{\tilde{n}}))^{n+n(n+1)/2},$$
$$A = (\partial/\partial x_1,\ldots, \partial/\partial x_n, \partial^2/\partial x_1\,\partial x_1,\ldots, \partial^2/\partial x_n\,\partial x_n),$$
$$l(u,v,p,q) = A(u,v) - (p,q)$$

for each $(u,v) \in Z$, $(p,q) \in Z_J$.

We assume that the operator on the right side of the equation is locally uniformly strongly elliptic, i.e., for each bounded set $B \subset R^{2\tilde{n}+m}$, there exists a constant $c = c(B) > 0$ such that

$$\left(\sum_{1 \leqslant i \leqslant j \leqslant n} \frac{\partial a(x,u,v,p,q)}{\partial p^{i,j}}\,\xi_i \xi_j v, v\right) > c\,|\xi|^2\,|v|^2$$

for each $(x, u, v, p, q) \in R^n \times B$, $\xi \in R^n$, $v \in R^{\tilde{n}}$. For $(u, v, p, q) \in X \times Y_J$, define $g(u, v, p, q) \in H^{s-1}(R^n, R^{\tilde{n}})$ by $g(u, v, p, q)(x) = a(x, u(x), v(x), p(x))$, and define $f(u, v, p, q) \in Z$ by $f(u, v, p, q) = (v, g(u, v, p, q))$. For each $1 \leqslant i \leqslant n$, define $g_{x_i}(u, v, p, q)$ by $g_{x_i}(u, v, p, q)(x) = \partial a/\partial x_i\,(x, u(x), v(x), p(x))$, define $g_{p^i}(u, v, p, q)$ by $g_{p^i}(u, v, p, q)(x) = (\partial a/\partial p^i)(x, u(x), v(x), p(x))$, etc.

We first compute $A \circ f$. For each $1 \leqslant i \leqslant j \leqslant n$,

$$\lambda_i \circ A \circ f(u, v, p, q) = v_i,$$

$$\lambda_{i,j} \circ A \circ f(u, v, p, q) = v_{i,j},$$

$$\begin{aligned} v_i \circ A \circ f(u, v, p, q) &= g_{x_i}(u, v, p, q) + g_u(u, v, p, q)\, u_i + g_v(u, v, p, q)\, v_i \\ &\quad + g_{p^k}(u, v, p, q)(p^k)_i + g_{q^k}(u, v, p, q)(q^k)_i + g_{p^{k_1,k_2}}(u, v, p, q)(p^{k_1,k_2})_i, \end{aligned}$$

$$\begin{aligned} v_{i,j} \circ A \circ f(u, v, p, q) &= g_{x_i,x_j}(u, v, p, q) + g_{u,x_i}(u, v, p, q)\, u_j + g_{u,x_j}(u, v, p, q) u_i \\ &\quad + g_{u,u}(u, v, p, q)(u_i, u_j) + g_{v,x_i}(u, v, p, q)\, v_j + g_{v,x_j}(u, v, p, q)\, v_i \\ &\quad + g_{v,v}(u, v, p, q)(v_i, v_j) + g_{u,v}(u, v, p, q)((u_i, v_j) + (u_j, v_i)) \\ &\quad + g_{p^k,x_i}(u, v, p, q)(p^k)_j + g_{p^k,x_j}(u, v, p, q)(p^k)_i \\ &\quad + g_{p^k,u}(u, v, p, q)(((p^k)_i, u_j) + ((p^k)_j, u_i)) \\ &\quad + g_{p^k,v}(u, v, p, q)(((p^k)_i, v_j) + ((p^k)_j, v_i)) \\ &\quad + g_{q^k,u}(u, v, p, q)(((q^k)_i, u_j) + ((q^k)_j, u_i)) \\ &\quad + g_{q^k,v}(u, v, p, q)(((q^k)_i, v_j) + ((q^k)_j, v_i)) \\ &\quad + g_{q^k,x_i}(u, v, p, q)(q^k)_j + g_{q^k,x_j}(u, v, p, q)(q^k)_i \\ &\quad + g_{p^{k_1},p^{k_2}}(u, v, p, q)((p^{k_1})_i, (p^{k_2})_j) + g_{q^{k_1},q^{k_2}}(u, v, p, q)((q^{k_1})_i, (q^{k_2})_j) \\ &\quad + g_{p^{k_1},q^{k_2}}(u, v, p, q)(((p^{k_1})_i, (q^{k_2})_j) + ((p^{k_1})_j, (q_{k_2})_i)) \\ &\quad + g_{p^{k_1,k_2},x_i}(u, v, p, q)(p^{k_1,k_2})_j + g_{p^{k_1,k_2},x_j}(u, v, p, q)(p^{k_1,k_2})_i \\ &\quad + g_{p^{k_1,k_2},u}(u, v, p, q)(((p^{k_1,k_2})_i, u_j) + ((p^{k_1,k_2})_j, u_i)) \\ &\quad + g_{p^{k_1,k_2},v}(u, v, p, q)(((p^{k_1,k_2})_i, v_j) + ((p^{k_1,k_2})_j, v_i)) \\ &\quad + g_{p^{k_1,k_2},p^{k_3}}(u, v, p, q)(((p^{k_1,k_2})_i, (p^{k_3})_j) + ((p^{k_1,k_2})_j, (p^{k_3})_i)) \\ &\quad + g_{p^{k_1,k_2},q^{k_3}}(u, v, p, q)(((p^{k_1,k_2})_i, (q^{k_3})_j) + ((p^{k_1,k_2})_j, (q^{k_3})_i)) \\ &\quad + g_{p^{k_1,k_2},p^{k_3,k_4}}(u, v, p, q)((p^{k_1,k_2})_i, (p^{k_3,k_4})_j) + g_u(u, v, p, q) u_{i,j} \end{aligned}$$

$$+ g_v(u, v, p, q)\, v_{ij} + g_{p^k}(u, v, p, q)(p^k)_{i,j}$$
$$+ g_{q^k}(u, v, p, q)(q^k)_{i,j} + g_{p^{k_1,k_2}}(u, v, p, q)(p^{k_1,k_2})_{i,j}.$$

Define h: $X \times Y_J \to Z_J$ as follows:

$$\lambda_i \circ h(u, v, p, q) = q^i,$$
$$\lambda_{i,j} \circ h(u, v, p, q) = q^{i,j},$$
$$v_i \circ h(u, v, p, q) = v_i \circ A \circ f(u, v, p, q) + g_{p^{k_1,k_2}}(u, v, p, q)((p^i)_{k_1,k_2} - (p^{k_1,k_2})_i),$$
$$\begin{aligned} v_{i,j} \circ h(u, v, p, q) = {} & v_{i,j} \circ A \circ f(u, v, p, q) \\ & + g_u(u, v, p, q)(p^{i,j} - u_{i,j}) + g_v(u, v, p, q)(q^{i,j} - v_{i,j}) \\ & + g_{p^k}(u, v, p, q)((p^{i,j})_k - (p^k)_{i,j}) \\ & + g_{q^k}(u, v, p, q)((q^{i,j})_k - (q^k)_{i,j}) \\ & + g_{p^{k_1,k_2}}(u, v, p, q)((p^{i,j})_{k_1,k_2} - (p^{k_1,k_2})_{i,j}). \end{aligned}$$

Define $\tilde{f}$: $X \times Y_J \to Z$ by

$$\lambda_0 \circ \tilde{f}(u, v, p, q) = v,$$
$$\begin{aligned} v_0 \circ \tilde{f}(u, v, p, q) = {} & g(u, v, p, q) + g_{p^k}(u, v, p, q)(u_k - p^k) \\ & + g_{q^k}(u, v, p, q)(v_k - q^k) \\ & + g_{p^{k_1,k_2}}(u, v, p, q)(u_{k_1,k_2} - p^{k_1,k_2}), \end{aligned}$$

where λ_0 and v_0 are the first and second coordinate projections on Z, respectively. Define $H(\cdot, \cdot, \cdot, \cdot)$ by $H(u, v, p, q) = (\tilde{f}(u, v, p, q), h(u, v, p, q))$. Then $H(\cdot, \cdot, \cdot, \cdot)$ satisfies the conditions of the quasilinear equation derived in [6, Sect. 3.2]. It follows from [6] that $H(\cdot, \cdot, \cdot, \cdot)$ generates a local flow on $X \times Y_J$.

We next compute $A \circ \tilde{f} - h$:

$$\lambda_i(A \circ \tilde{f}(u, v, p, q) - h(u, v, p, q)) = v_i - q^i,$$
$$\lambda_{i,j}(A \circ \tilde{f}(u, v, p, q) - h(u, v, p, q)) = v_{i,j} - q^{i,j},$$
$$\begin{aligned} & v_i(A \circ \tilde{f}(u, v, p, q) - h(u, v, p, q)) \\ & \quad = g_{p^{k_1,k_2}}(u, v, p, q)(u_{k_1,k_2,i} - (p^{k_1,k_2})_i) \\ & \qquad + \frac{\partial}{\partial x_i}(g_{p^{k_1,k_2}}(u, v, p, q))(u_{k_1,k_2} - p^{k_1,k_2}) \end{aligned}$$

$$
\begin{aligned}
&+\frac{\partial}{\partial x_i}(g_{p^k}(u,v,p,q))(u_k-p^k)\\
&+g_{p^k}(u,v,p,q)\frac{\partial}{\partial x_i}(u_k-p^k)+g_{q^k}(u,v,p,q)\frac{\partial}{\partial x_i}(v_k-q^k)\\
&+\frac{\partial}{\partial x_i}(g_{q^k}(u,v,p,q))(v_k-q^k)\\
&-g_{p^{k_1,k_2}}(u,v,p,q)((p^i)_{k_1,k_2}-(p^{k_1,k_2})_i)\\
={}&g_{p^{k_1,k_2}}(u,v,p,q)\frac{\partial^2}{\partial x_{k_1}\partial x_{k_2}}(u_i-p^i)\\
&+g^{i,1}(u,v,p,q)(A(u,v)-(p,q))\\
&+g^{i,0}(u,v,p,q)\frac{\partial}{\partial x_i}(A(u,v)-(p,q)),
\end{aligned}
$$

where $g^{i,1}(\cdot,\cdot,\cdot,\cdot)$ and $g^{i,0}(\cdot,\cdot,\cdot,\cdot)$ are polynomial differential operators of weight and order 1 and 0, respectively, in the derivatives of u, v, p, q. Similarly,

$$
\begin{aligned}
&\nu_{i,j}(A\circ\tilde{f}(u,v,p,q)-h(u,v,p,q))\\
&\quad=g_{p^{k_1,k_2}}(u,v,p,q)\frac{\partial^2}{\partial x_{k_1}\partial x_{k_2}}(u_{i,j}-p^{i,j})\\
&\quad\quad+g^{i,j,2}(u,v,p,q)(A(u,v)-(p,q))\\
&\quad\quad+g^{i,j,1,k}(u,v,p,q)\frac{\partial}{\partial x_k}(A(u,v)-(p,q)),
\end{aligned}
$$

where $g^{i,j,2}(\cdot,\cdot,\cdot,\cdot)$ and $g^{i,j,1,k}(\cdot,\cdot,\cdot,\cdot)$ are polynomial differential operators of weight and order 2 and 1, respectively, in the derivatives of u, v, p, q.

For each $(u,v,p,q)\in Z\times Z_J$, $(\bar{p},\bar{q})\in V_1$, define $F(u,v,p,q)(\bar{p},\bar{q})\in V_2$ by

$$
\begin{aligned}
&\lambda_i(F(u,v,p,q)(\bar{p},\bar{q}))=\bar{q}^i,\\
&\lambda_{i,j}(F(u,v,p,q)(\bar{p},\bar{q}))=\bar{q}^{i,j},\\
&\nu_i(F(u,v,p,q)(\bar{p},\bar{q}))=g_{p^{k_1,k_2}}(u,v,p,q)\frac{\partial^2\bar{p}^i}{\partial x_{k_1}\partial x_{k_2}}\\
&\qquad\qquad+g^{i,0}(u,v,p,q)(\bar{p},\bar{q})\\
&\qquad\qquad+g^{i,1}(u,v,p,q)\frac{\partial}{\partial x_i}(\bar{p},\bar{q}),
\end{aligned}
$$

$$v_{i,j}(F(u, v, p, q)(\bar{p}, \bar{q})) = g_{pk_1,k_2}(u, v, p, q) \frac{\partial^2 \bar{p}^{i,j}}{\partial x_{k_1} \partial x_{k_2}} + g^{i,j,2}(u, v, p, q)(\bar{p}, \bar{q}) + g^{i,j,1,k}(u, v, p, q) \frac{\partial}{\partial x_k} (\bar{p}, \bar{q}).$$

Let $\sigma(\cdot)$ be an integral curve for $H(\cdot, \cdot)$, and let (ε, δ) be the domain of $\sigma(\cdot)$, where $\varepsilon < 0 < \delta$. Since $\sigma(\cdot)$ is continuously differentiable in $(H^{s-1}(R^n, R^{\tilde{n}}) \times H^{s-2}(R^n, R^{\tilde{n}}))^{1+n+n(n+1)/2}$ and $s - 4 > n/2$, the differential equation $\dot{\rho}(t) = F(\sigma(t))\,\rho(t)$ is a time-dependent linear wave equation in (V_1, V_2) which satisfies the hypotheses of [7, Theorem 6.1]. Thus this equation generates a linear evolution system $\{U(t, s) \in L(V_2, V_2)$: $\varepsilon < s$, $t < \delta\}$, which implies that the only solution to the initial-value problem $\dot{\rho}(t) = F(\sigma(t))\,\rho(t)$, $\rho(0) = 0$, is the curve $\rho(\cdot) \equiv 0$.

EXAMPLE. For $\tilde{n} = 1$, consider the equation

$$\frac{\partial^2 u}{\partial t^2} = e^{\Delta u} - 1, \qquad \text{where} \quad \Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}.$$

Remark. A generalization of the above argument could be used to prove existence of local flows and uniqueness of integral curves for nonlinear wave equations of order $2m$ for arbitrary $m \in N$. However, it would first be necessary to verify that the quasilinear theory in [6] generalizes to higher order quasilinear wave equations.

3.4. *Global Existence for a Generalized Korteweg–de Vries Equation*

Consider the generalized Korteweg–de Vries equation

$$\frac{\partial u}{\partial t}(t, x) = \frac{\partial^3 u}{\partial x^3}(t, x) + c\left(\frac{\partial u}{\partial x}(t, x)\right)^2 \qquad \text{in} \quad R \times R,$$

where $c \in R$. If $c = 0$, the equation is linear and existence of a globally defined flow on all H^s-spaces is trivial, so assume $c \neq 0$. While this equation is quasilinear in the sense introduced by Sobolevskii [13], it is not parabolic. Thus Sobolevskii's quasilinear existence theory does not apply to this equation. Since the equation is not quasilinear in the more restrictive sense introduced by Kato [9], Kato's quasilinear existence theory does not apply either. Nonetheless, existence of a flow for this equation can easily be established via the correspondence theorem.

Let $s > \frac{3}{2}$, $Y_J = H^s(R, R)$, $X = Z = H^{s-2}(R, R)$, $Y = H^{s+1}(R, R)$, $Z_J = H^{s-3}(R, R)$, $A = \partial/\partial x$. Define f: $X \times Y_J \to Z$ by $f(u, p) = p_{xx} + cp^2$. Then g:

$Y \to Z$ is given by $g(u) = f \circ (Id \times A)(u) = u_{xxx} + c(u_x)^2$, and $A \circ f(u, p) = p_{xxx} + 2cpp_x$.

Define $h = A \circ f$ and $\tilde{f} = f$. Since $A \circ \tilde{f} = h$, we may let $V_1 = V_2 = Z_J$ and $F(\cdot, \cdot) \equiv 0$ and conclude that solutions to $g(\cdot)$ and $H(\cdot, \cdot)$ are in one-to-one correspondence.

Now, the equation $\dot{p} = h(u, p) = p_{xxx} + 2cpp_x$ is just the classical Korteweg–de Vries equation. It was shown by Kato in [10] that this equation generates a local flow on $H^s(R, R)$, and a global flow if $s \geqslant 2$. Let $(y_0, p_0) \in X \times Z$ and let $\sigma(\cdot)$ be the maximally defined integral curve for the KdV equation with initial value p_0. Define the curve $\gamma(\cdot)$ by $\gamma(t) = (\sigma(t))_{xx} + c(\sigma(t))^2$. Since $\gamma(\cdot)$ is a continuous curve in X, the initial value problem $\dot{u}(t) = \gamma(t)$, $u(0) = y_0$, has a unique continuously differentiable solution $\rho(\cdot)$ in X which is given by

$$\rho(t) = y_0 + \int_0^t \gamma(r)\, dr.$$

Clearly $\rho(\cdot)$ is defined on the same interval as $\sigma(\cdot)$ and depends continuously in X on (y_0, p_0). Thus $H(\cdot, \cdot)$ generates a local flow on $X \times Y_J$ for $s > \frac{3}{2}$ and a global flow for $s \geqslant 2$. By the correspondence theorem, we conclude that the equation $\dot{u} = u_{xxx} + c(u_x)^2$ generates a local flow on $H^{s+1}(R, R)$ for $s > \frac{3}{2}$ and a global flow for $s \geqslant 2$.

Remark. By using a slightly more complicated method of proof, we can prove local existence and uniqueness for the general nonlinear KdV equation. Let $a(\cdot, \cdot)$: $R \times R^2 \to R$ be a type $\mathscr{S}$ map, and consider the equation

$$\frac{\partial u}{\partial t}(t, x) = \frac{\partial^3 u}{\partial x^3}(t, x) + a\left(x, u(t, x), \frac{\partial u}{\partial x}(t, x)\right) \qquad \text{in} \quad R \times R.$$

Let $s \geqslant 2$ (we will reduce this to $s > \frac{3}{2}$ in the last part of the proof), $X = Y_J = H^s(R, R)$, $Y = H^{s+1}(R, R)$, $Z = H^{s-3}(R, R)$, $Z_J = H^{s-4}(R, R)$, $V_1 = H^{s-1}(R, R)$, $V_2 = H^{s-4}(R, R)$, $A = \partial/\partial x$, $l(u, p) = Au - p$.

Define g: $X \times Y_J \to^{C^\infty} X$ by $g(u, p)(x) = a(x, u(x), p(x))$, and define f: $X \times Y_J \to Z$ by $f(u, p) = u_{xxx} + g(u, p)$. Then $A \circ f(u, p) = u_{xxxx} + g_x(u, p) + g_u(u, p)\, u_x + g_p(u, p)\, p_x$. So define $h(u, p) = p_{xxx} + g_x(u, p) + g_u(u, p)\, p + g_p(u, p)\, p_x$, and let $\tilde{f} = f$. Then

$$\begin{aligned} H(u, p) &= (u_{xxx} + g(u, p), p_{xxx} + g_x(u, p) + g_u(u, p)\, p + g_p(u, p)\, p_x) \\ &= (u, p)_{xxx} + g_p(u, p)(0, p)_x + (g(u, p), g_x(u, p) + g_u(u, p)\, p). \end{aligned}$$

From the form of the expression for $H(\cdot, \cdot)$, it is easy to see that Kato's treatment of existence theory for generalized KdV equations in [10] applies with only minor modification to show that $H(\cdot, \cdot)$ generates a local flow in $X \times Y_J$. It also follows easily that $A \circ \tilde{f}(u, p) - h(u, p) = ((\partial^3/\partial x^3) + g_u(u, p))(Au - p)$. Since multiplication by each $g_u(u, p)$ induces a bounded linear operator on $V_2 = H^{s-4}(R, R)$ (this is the reason we need to assume $s \geqslant 2$), it follows that for each continuous curve $\gamma(\cdot)$ in $X \times Y_J$, the time-dependent linear equation $\dot{\rho}(t) = F(\gamma(t))\rho(t)$ generates a linear evolution system on V_2, where each $F(u, p) \in L(V_1, V_2)$ is defined by $F(u, p)(\bar{p}) = \bar{p}_{xxx} + g_u(u, p)\bar{p}$. Uniqueness of the solution to the initial value problem $\dot{\rho}(t) = F(\gamma(t))\rho(t)$, $\rho(0) = 0$, follows immediately. Thus the correspondence theorem implies that the equation $\dot{u} = u_{xxx} + g(u, u_x)$ generates a local flow in Y.

Now assume that $\frac{3}{2} < s < 2$. Kato's local existence theory [10] still applies to show that $H(\cdot, \cdot)$ generates a local flow in $X \times Y_J$. To show that this flow induces a local flow in Y for our nonlinear equation, it suffices to show that $Au(\cdot) - p(\cdot) \equiv 0$ for each integral curve in $X \times Y_J$.

For each $(u_0, p_0) \in H^2(R, R) \times H^2(R, R)$, the integral curve $\sigma(\cdot) = (u(\cdot), p(\cdot))$ in $X \times Y_J$ with initial value (u_0, p_0) is a continuous curve in $H^2(R, R) \times H^2(R, R)$ as long as it remains a continuous curve in $X \times Y_J$ (by [10, Theorem 1(c)]). Thus, for such integral curves, $Au(\cdot) - p(\cdot) \equiv 0$. Since $H^2(R, R) \times H^2(R, R)$ is dense in $X \times Y_J$, and since integral curves in $X \times Y_J$ depend continuously on the initial data, it follows that $Au(\cdot) - p(\cdot) \equiv 0$ for all integral curves $\sigma(\cdot)$ in $X \times Y_J$ with no restriction on the initial data. Thus we conclude that there exists a local flow in $H^{s+1}(R, R)$ for the equation $\dot{u} = u_{xxx} + g(u, u_x)$ for $s > \frac{3}{2}$, and that the integral curves are unique.

4. Vector Fields on Manifolds of Maps

It has been known for many years that nonlinear parabolic and hyperbolic partial differential equations can be regarded formally as unbounded vector fields in Banach function spaces. However, lack of a nonlinear existence theory for integral curves of such vector fields has prevented formal incorporation of the theory of evolution equations into global analysis.

The present article and the papers of Dorroh [1, 2] should remove most major obstacles to globalization of evolution equations. The significant observation to make in this regard concerning applications treated in [1] and in the present article is that the critical condition on each equation which guarantees existence of a local semiflow is usually a condition on the linearization of the equation with respect to the highest order partial derivatives. These conditions can be rephrased more abstractly as assumptions about the symbols of the differential operators (often with respect to some metric).

The notion of a nonlinear differential operator has been extended by

Uhlenbeck to the notion of a differential section operator on a manifold of maps [14, Sect. 6]. Differential section operators induce unbounded vector fields on manifolds of maps. Since differential section operators have well-defined symbols (for a general discussion of the symbols of nonlinear differential operators, see [11, Chap. 17]), we may use this concept to reformulate the second and third applications of this article in terms of the existence of local (semi)flows for certain types of unbounded vector fields on manifolds of maps between finite-dimensional Riemannian manifolds. The symmetric first-order hyperbolic existence theory developed in [1] may also admit such a globalized version, and other existence theorems on integral curves for unbounded vector fields on manifolds of maps will undoubtedly be developed. A fusion of the methods and results of evolution equations and global analysis seems likely to follow. It is hoped that this will result in new and interesting applications to the theory of nonlinear partial differential equations.

References

1. J. R. Dorroh, Nonlinear symmetric first-order systems, *J. Math. Anal. Appl.* **91** (1983), 523–526.
2. J. R. Dorroh, "Nonlinear Parabolic Systems," preprint.
3. J. R. Dorroh and R. Graff, Integral equations in Banach spaces, a general approach to the linear Cauchy problem, and applications to the nonlinear problem, *J. Integral Equations* **1** (1979), 309–359.
4. R. Graff, "A Functional Analytic Approach to Uniqueness of Solutions to Some Nonlinear Cauchy Problems," preprint.
5. R. Graff, "Well-Posedness of Nonlinear Parabolic Cauchy Problems," in preparation.
6. T. Hughes, T. Kato, and J. Marsden, Well-posed quasi-linear second-order hyperbolic systems with applications to nonlinear elastodynamics and general relativity, *Arch. Rational Mech. Anal.* **63** (1977), 273–294.
7. T. Kato, Linear evolution equations of "hyperbolic" type, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **17** (1970), 241–258.
8. T. Kato, The Cauchy problem for quasi-linear symmetric hyperbolic systems, *Arch. Rational Mech. Anal.* **58** (1975), 181–205.
9. T. Kato, Quasi-linear equations of evolution, with applications to partial differential equations, Lecture Notes in Mathematics No. 448 pp. 25–70, Springer–Verlag, Berlin/New York, 1975.
10. T. Kato, On the Korteweg–de Vries equation, *Manuscripta Math.* **28** (1979), 89–100.
11. R. S. Palais, "Foundations of Global Non-Linear Analysis," Benjamin, New York, 1968.
12. R. S. Phillips, Dissipative operators and hyperbolic systems, *Trans. Amer. Math. Soc.* **90** (1959), 193–254.
13. P. E. Sobolevskii, Equations of parabolic type in a Banach space, *Amer. Math. Soc. Transl.* (2) **49** (1965), 1–62.
14. K. Uhlenbeck, "The Calculus of Variations and Global Analysis," Ph.D. Dissertation, Brandeis University, Waltham, Massachusetts, 1968.